Discussion 20 Worksheet
More Stokes and Divergence
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MATH 53 Multivariable Calculus

1 Archimedes' Principle

The Force exerted on a solid D with surface S fully submerged in water is given by

$$\mathbf{F} = \iint_S -p\hat{\mathbf{n}}\, dS$$

where the pressure p is given by $-\rho g z$ (ρ is the density of water and g the gravitational acceleration) if we assume that the surface of the water is at $z = 0$. Use the divergence theorem to show Archimedes' principle $\vec{F} = \rho g \,\mathrm{vol}(D)$.
Hint: Compute $\mathbf{F} \cdot \mathbf{i}, \mathbf{F} \cdot \mathbf{j}, \mathbf{F} \cdot \mathbf{k}$

2 identities

Prove each of these identities, assuming that D is a solid region in 3D space and $S = \partial D$.

(i) $\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$ where $\mathbf{a}$ is any constant vector.

(ii) $\mathrm{Vol}(D) = \frac{1}{3} \iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x, y, z)$

(iii) $\iint_S (f\nabla g)\, dS = \iiint_D f\nabla^2 g - \nabla f \cdot \nabla g\, dV$

3 parametrizing surfaces

Parametrize the following surfaces

(a) $x^2 + y^2 + 1 = z^2$ *(Hint: graph)*

(b) $x^2 + y^2 = z^2$ *(try spherical coordinates)*

(c) $x^2 + y^2 = (1 + z^2)^2$ *(Hint: cylindrical coordinates)*

(d) $y^2 + z^2 = e^x$

(e) $e^x = 1 + y^2 + 2\cos^2 z$

(f) $(x^2 + y^2 + z^2)^{3/2} = 2x^2 + 2y^2 + z^2$

(g) $x \sin z = y \cos z$

4 Stokes / surface integrals

1. Compute $\iint_S \vec{F} \cdot d\mathbf{S}$ where $\vec{F} = (x^2, 2z, -3y)$ and S is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$.

2. Compute $\iint_S (\nabla \times \vec{F}) \cdot d\mathbf{S}$ where $\vec{F} = (y, -x, yx^3)$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upwards orientation.

3. Compute $\iint_S \vec{F} \cdot d\mathbf{S}$ where $\vec{F} = (\sin(\pi x), zy^3, z^2 + 4x)$ where S is the surface of the box $-1 \leq x \leq 2$, $0 \leq y \leq 1$, and $1 \leq z \leq 4$, oriented outwards.

5 Stokes' theorem

1. Verify Stokes' theorem for the following surfaces S and vector fields $\mathbf{F}$.

(a) $\mathbf{F}(x, y, z) = (y, z, x)$, S is the hemisphere $x^2 + y^2 + z^2 = 1, y \geq 0$.

(b) $\mathbf{F}(x, y, z) = (-y, x, -2)$ and S is the cone $z^2 = x^2 + y^2, 0 \leq z \leq 4$.

2. Use Stokes theorem to compute the integral of $\mathbf{F}(x, y, z) = (-y + ze^{x^2-y^2}, x + ze^{y^2-x^2}, 0)$ around the circle $x^2 + y^2 = 1, \underline{z = 1}$ *(Hint: The integral would be easy if $z = 0$.)*

6 Past final problems

1. Let C be the spiral $r = \theta$ between $\theta = 0$ and $\theta = a$, for some $a > 0$.

   a) Set up an integral to find the integral of xy over C with respect to arc length. Do not attempt to evaluate the integral.

   b) Write down a vector field $\vec{F}$ (not depending on a) such that $\int_C \vec{F} \cdot d\vec{r}$ is equal to the integral in (a).

2. Calculate $\iint_S \vec{F} \cdot d\mathbf{S}$ where S is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented using the outward pointing normal, and

$$\vec{F} = (x + \sin y, y + \sin z, z + \sin x).$$

3. Let $\vec{r}_1$ and $\vec{r}_2$ be two parametric curves in three dimensions that satisfy

$$\frac{d\vec{r}_1}{dt} = \vec{r}_2 - \vec{r}_1$$
$$\frac{d\vec{r}_2}{dt} = \vec{r}_2 + \vec{r}_1.$$

   Show that $\vec{r}_1 \times \vec{r}_2$ is constant in time.

4. Find the volume of the solid enclosed by the surface

$$(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2).$$